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Short Communication

Harmonic balance and iteration calculations of periodic solutions to $\ddot{y} + y^{-1} = 0$

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Abstract

We use two methods, harmonic balance and iteration, to calculate analytical approximations to the periodic solutions of a nonlinear singular oscillator.

An important and interesting nonlinear differential equation is the following one:

$$\ddot{y} + \frac{c}{y} = 0, \quad c > 0. \tag{1}$$

This equation occurs in the modeling of certain phenomena in plasma physics [1]. An examination of the physical principles behind the derivation of the equation [1], along with a study of the $(y, v = \dot{y})$ phase-plane [2] shows that all solutions are periodic. Using methods given in Mickens [2], it can be shown that the exact value for the angular frequency is given by the expression

$$\omega_{ex}(A) = \frac{2\pi}{T_{ex}(A)}, \tag{2}$$

where the period is

$$T_{ex}(A) = 2\sqrt{2}A \int_0^1 \frac{ds}{\sqrt{\ln(1/s)}}, \tag{3}$$

and the following initial conditions were selected:

$$y(0) = A, \quad \dot{y}(0) = 0 \tag{4}$$

with $c = 1$. In the calculations to come, this value is always used, i.e., Eq. (1) becomes

$$\ddot{y} + \frac{1}{y} = 0. \tag{5}$$

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The definite integral in Eq. (3) can be easily evaluated [4]; doing this gives [1,3]

$$T_{\mathrm{ex}}(A) = 2\sqrt{2\pi}\,A, \tag{6}$$

or

$$\omega_{\mathrm{ex}}(A) = \frac{\sqrt{2\pi}}{2A} = \frac{1.2533141}{A}. \tag{7}$$

The purpose of this Short Communication is to calculate analytical approximations to the periodic solutions of Eq. (5) using both harmonic balance methods and an iteration scheme. The results obtained provide excellent approximations to the exact solution as measured by the percentage error in the calculated values for the angular frequency in comparison to the exact value of Eq. (7).

The first-order harmonic balance solution takes the form [2]

$$y_0(t) = A\cos(\omega t). \tag{8}$$

Observe that $y_0(t)$ satisfies the initial conditions, Eq. (4), and the task is to calculate $\omega(A)$. At this point, the essential question is how to rewrite Eq. (5) such that the method of harmonic balance can be easily applied. The most convenient way to do this is to express it as

$$y\ddot{y} + 1 = 0. \tag{9}$$

Substitution of Eq. (8) into Eq. (9) gives

$$\left[-\left(\frac{\omega^2 A^2}{2}\right) + 1\right](\mathrm{HOH}) \simeq 0, \tag{10}$$

where HOH $\equiv$ higher-order harmonics. Thus, in the lowest-order it follows that

$$-\left(\frac{\omega^2 A^2}{2}\right) + 1 = 0 \tag{11}$$

and

$$\omega_0(A) = \frac{\sqrt{2}}{A} = \frac{1.4142}{A}. \tag{12}$$

The next level of harmonic balance uses the form

$$y_1(t) = A_1\cos\theta + A_2\cos 3\theta, \quad \theta = \omega t, \tag{13}$$

where (A_1, A_2, ω) are to be determined as functions of A. Substituting Eq. (13) into Eq. (9), carrying out the various trigonometric operations, the following result is obtained:

$$\left[-\omega^2\left(\frac{A_1^2 + 9A_2^2}{2}\right) + 1\right] - \omega^2\left(\frac{A_1^2 + 10A_1A_2}{2}\right)\cos 2\theta + \mathrm{HOH} \simeq 0. \tag{14}$$

Setting the constant term and the coefficient of $\cos 2\theta$ equal to zero gives two algebraic equations

$$-\omega^2\left(\frac{A_1^2 + 9A_2^2}{2}\right) + 1 = 0, \quad \left(\frac{\omega^2 A_1}{2}\right)(A_1 + 10A_2) = 0. \tag{15}$$

The nontrivial solution to the second equation is $A_2 = -(A_1/10)$. Substitution of this into the first equation gives

$$\omega_1^2(A_1) = \frac{200}{109A_1^2}. \tag{16}$$

With these values for A_1 and A_2, Eq. (13) can be written

$$y_1(t) = A_1\left[\cos(\omega_1 t) - \left(\frac{1}{10}\right)\cos(3\omega_1 t)\right]. \tag{17}$$

Requiring $y_1(0) = A$ gives

$$A_1 = \left(\frac{10}{9}\right)A, \tag{18}$$

and finally

$$y_1(t) = \left(\frac{10}{9}\right)A\left[\cos(\omega_1 t) - \left(\frac{1}{10}\right)\cos(3\omega_1 t)\right] \tag{19}$$

with

$$\omega_1^2(A) = \left(\frac{162}{100}\right)\frac{1}{A^2} \tag{20}$$

or

$$\omega_1(A) = \frac{1.272792}{A}. \tag{21}$$

Note that the percentage error is very small, i.e.,

$$\text{percentage error} \equiv \left|\frac{\omega_{\text{ex}} - \omega_1}{\omega_{\text{ex}}}\right| \cdot 100 = 1.6\%.$$

Iteration schemes provide an alternative procedure for calculating approximations to periodic solutions [3,5]. The scheme to now be used for Eq. (9) is

$$\ddot{y}_{k+1} + \Omega^2 y_{k+1} = \Omega^2 y_k - (\ddot{y}_k)^2 y_k, \tag{22}$$

where $k = 0, 1, 2, ...$; each $y_{k+1}(t)$ is required to satisfy

$$y_{k+1}(0) = A, \quad \dot{y}_{k+1}(0) = 0, \tag{23}$$

and at each stage k of the iteration the angular frequency is determined by the requirement that no secular terms appear in the solution for $y_{k+1}(t)$. The details as to how to apply these types of calculational schemes are given in Refs. [5,3]. This iteration procedure was formulated as follows:

$$y\ddot{y} + 1 = 0,$$

$$\ddot{y} + (\ddot{y})^2 y = 0,$$

$$\ddot{y} + \Omega^2 y = \Omega^2 y - (\ddot{y})^2 y \tag{24}$$

with, finally, the result

$$\ddot{y}_{k+1} + \Omega_{k+1}^2 y_{k+1} = \Omega_{k+1}^2 y_k - (\ddot{y}_k)^2 y_k. \tag{25}$$

The iteration starts by using for $y_0(t)$ the form

$$y_0(t) = A\cos\theta, \quad \theta = \Omega t. \tag{26}$$

Substitution of this into Eq. (25) gives

$$\ddot{y}_1 + \Omega_1^2 y_1 = \Omega_1^2 A\left(1 - \frac{3A^2\Omega_1^2}{4}\right)\cos\theta - \left(\frac{A^3\Omega_1^4}{4}\right)\cos 3\theta. \tag{27}$$

The absence of secular terms in $y_1(t)$ requires that the coefficient of the $\cos\theta$ term be zero. Using this, it follows that:

$$\Omega_1(A) = \frac{2}{\sqrt{3}A} = \frac{1.15470}{A}, \tag{28}$$

a value also obtained by He [3]. Thus, $y_1(t)$ is the solution to

$$\ddot{y}_1 + \Omega_1^2 y_1 = -\left(\frac{A^3\Omega_1^4}{4}\right)\cos 3\theta. \tag{29}$$

The complete solution to Eq. (29), satisfying $y_1(0) = A$ and $\dot{y}_1(0) = 0$ is

$$y_1(t) = A\cos\theta - \left(\frac{A^3\Omega_1^2}{32}\right)(\cos\theta - \cos 3\theta). \tag{30}$$

If $\Omega(A)$ is replaced by Eq. (28), then

$$y_1(t) = A\left[\left(\frac{23}{24}\right)\cos\left(\frac{2t}{\sqrt{3}A}\right) + \left(\frac{1}{24}\right)\cos\left(\frac{6t}{\sqrt{3}A}\right)t\right]. \tag{31}$$

Note that this calculation for $\Omega_1(A)$ has the following percentage error:

$$\left|\frac{\omega_{ex} - \Omega_1}{\omega_{ex}}\right| \cdot 100 = 7.9\%. \tag{32}$$

To calculate $y_2(t)$, the form of $y_1(t)$ given by Eq. (30) must be used and the differential equation to be solved is

$$\ddot{y}_2 + \Omega_2^2 y_2 = \Omega_2^2 y_1 - (\ddot{y}_1)^2 y_1. \tag{33}$$

The absence of secular terms for the solution of $y_2(t)$ will allow the determination of $\Omega_2(A)$. However, the work required to solve for $y_2(t)$ is very algebraic intensive and, as a consequence, $y_2(t)$ will not be given.

By comparison, the second-order harmonic balance method gives a more accurate solution than the first-order iteration procedure, i.e., 1.6% versus 7.9% for the calculated angular frequency. What the work of this paper demonstrates is that even for singular differential equations, such as Eq. (1), a variety of techniques exist for the determination of analytical approximations to the periodic solutions. (Eq. (1) is a singular differential equation because the second term on the left-side is not defined when $y = 0$.) Here, we have shown that the harmonic balance method provides an excellent approximation and that calculating this solution was quite easy. However, there exist other situations for which harmonic balance is not applicable and iteration procedures must be used [6].

The oscillator of Eq. (1) has also been studied by He [3] within the context of the homotopy perturbation method. For purposes of comparison with the results presented here, a summary of He's calculations is given below. Note that the notation of He's paper has been changed to conform with this paper's representations of equations and solutions.

He gives three approximations to the solutions of Eq. (1) or (5), with $c = 1$, using various formulations of the homotopy perturbation method. (See the introduction to Section 5 of He [3] for a detailed explanation of this method.) In the first approach, He obtains

$$y(t) \simeq A\cos(\omega t) + \left(\frac{1}{\omega}\right)\int_0^t \sin[\omega(s-t)]\left\{A\omega^2\cos(\omega s) - \frac{1}{A\cos(\omega s)}\right\}\mathrm{d}s, \tag{34}$$

with

$$\omega = \frac{\sqrt{2}}{A}. \tag{35}$$

In spite of the fact that the estimate for ω is reasonable, this solution must be rejected since an evaluation of the integral in Eq. (34) shows that it contains secular terms, i.e., solution that increase in time and/or can become unbounded; see Mickens [2] for a discussion of secular terms.

In the second approach, He finds that $y(t)$ is approximated by the following expressions:

$$y(t) \simeq A\cos(\omega t) + \left(\frac{A^3}{32}\right)[\cos(\omega t) - \cos(3\omega t)], \tag{36}$$

where

$$\omega = \frac{2}{\sqrt{3}A}. \tag{37}$$

Note that for this value of ω, the fractional error is 7.8%. However, this particular solution has a limitation on the values of the amplitude since the second term on the right-hand side of Eq. (36) should be smaller than the magnitude of the first term. This condition translates into the restriction $0<A<4$.

The third approach by He gives

$$y(t) \simeq A\left[\left(\frac{71}{72}\right)\cos(\omega t) - \left(\frac{1}{72}\right)\cos(3\omega t)\right], \tag{38}$$

with ω having the same value as that given in Eq. (37). Note that this approximation to the solution is more satisfactory than the previous two, the major reason being the smallness of the amplitude higher harmonic relative to the amplitude of the first harmonic.

In summary, He's results from the use of the homotopy perturbation method does not provide better approximations to the solutions of Eq. (1), with $c = 1$, than the calculations presented here using the direct second-order harmonic balance method for which the percentage error is very small, i.e., 1.6%. On the other hand, the iteration method of this paper gives exactly the same estimate for ω as He's second and third approaches, but has the advantages of having no limitations on the magnitude of the amplitude, A, and further gives a ratio for the amplitude of the third to first harmonic that is small, i.e., $1/23$.

The general conclusion is that the method of harmonic balance provides an easy and direct procedure for determining approximations to the periodic solutions of Eq. (1). This procedure also gives a very accurate estimate for ω at the second level of harmonic balancing. Finally, in comparison to the homotopy method, both harmonic balance and the iteration method produce better solutions.

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